Suggested Solutions for Quiz 1 of MATH 3270A

September, 2015

yxiao@math.cuhk.edu.hk

---

1. $y' + \left(\frac{3}{t}\right)y = \frac{\cos t}{t^3}$, $y(\pi)=0$, $t>0$.

Solution: suppose $\mu(t)$ is the integrating factor.

Then

$$\mu'(t) = \frac{3}{t}\mu(t) \quad / \quad \mu(t) = e^{\int \frac{3}{t}dt} \longleftarrow 3'$$

$$\Rightarrow \mu(t) = t^3 \longleftarrow 2'$$

Multiply $\mu(t)$ to the equation:

$$(t^3 y)' = \cos t$$

$$t^3 y(t) = \sin t + C \longleftarrow 3' \text{ (general solution)}$$

Substitute the initial condition:

$$\pi^3 \times y(\pi) = \sin\pi + C$$

$$\Rightarrow C = 0. \longleftarrow 2' \text{ (initial condition)}$$

$$y(t) = \frac{1}{t^3}\sin t. \quad (t>0)$$

#.

2. $y' = xy^3\sqrt{1+x^2}$, $y(0) = 1$.

a. Find the solution in explicit form;

b. Plot the graph of the solution;

c. Determine (at least approximately), the interval in which the solution is defined.

Solution:

a. $$\frac{dy}{y^3} = x\sqrt{1+x^2}\,dx$$ ← 3′ (seperable form)

integrate on both sides,

$$-\frac{1}{2}\frac{1}{y^2} = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$ ← 3′ (general solution)

Substitute the initial condition,

$$-\frac{1}{2}\frac{1}{y(0)^2} = \frac{1}{3}(1+0^2)^{\frac{3}{2}} + C$$

$$\Rightarrow C = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$$ ← 2′ (initial condition)

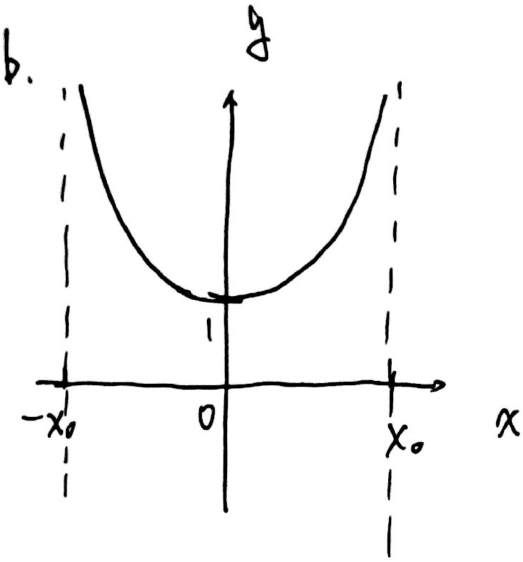
$$\Rightarrow \frac{1}{y^2} = \frac{5}{3} - \frac{2}{3}(1+x^2)^{\frac{3}{2}}$$

$$y = \frac{\pm 1}{\sqrt{\frac{5}{3} - \frac{2}{3}(1+x^2)^{\frac{3}{2}}}}$$

Since $y(0) = 1$,

$$y = \frac{1}{\sqrt{\frac{5}{3} - \frac{2}{3}(1+x^2)^{\frac{3}{2}}}}$$ ← 2′ (explicit solution)

b.



① $y(0)=1$

② $y(x)=y(-x)$

③ $y \uparrow$ on $(0, x_0)$

$y \downarrow$ on $(-x_0, 0)$

④ $y \uparrow +\infty$ as $x \to \pm x_0$

$2' \times 4 = 8'$

c. the solution is defined on

$$\frac{5}{3} - \frac{2}{3}(1+x^2)^{\frac{3}{2}} > 0.$$ (Could NOT equal to zero) $\leftarrow 4'$

$$\Rightarrow -\sqrt{\left(\frac{5}{2}\right)^{\frac{2}{3}}-1} < x < \sqrt{\left(\frac{5}{2}\right)^{\frac{2}{3}}-1}$$ $\leftarrow 3'$

#

---

3. Sketch f(y) versus y and determine critical points (or point) and classify each one asymptotically stable, unstable, or semistable.

25'

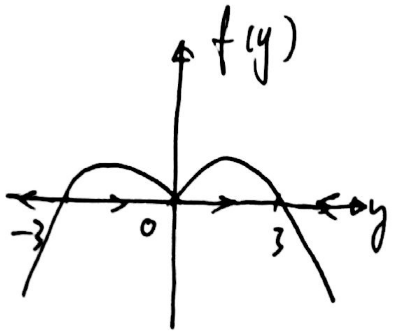
Draw the phase line and sketch several graphs of solutions in ty-plane.

$$\frac{dy}{dt} = y^2(9-y^2), \quad -\infty < y_0 < \infty.$$

Solution:

Critical points: $f(y) = y^2(9-y^2) = 0$.

$y_1 = 0 \quad y_2 = -3. \quad y_3 = +3$ ← $2' \times 3 = 6'$



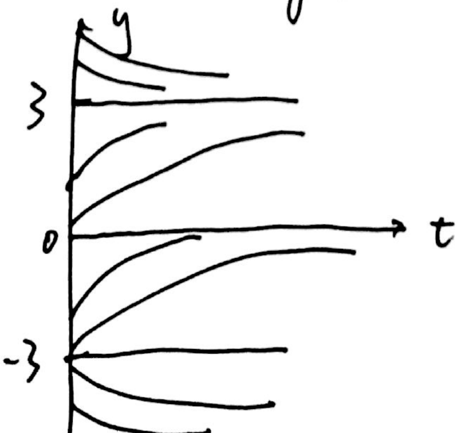
← 5'

Hence,

$y_1 = 0$, semistable,

$y_2 = -3$, unstable. ← $2' \times 3 = 6'$

$y_3 = 3$, asymptotically stable.



8'

4. Show that it is not exact but becomes exact by multiplying the given integrating factor and solve it

$$2x^2y^3 + x(1+y^2)y' = 0. \quad \mu = \frac{1}{xy^3}$$

$M(x,y) = 2x^2y^3$. $N(x,y) = x(1+y^2)$

$\partial_y M(x,y) = 6x^2y^2$. $\partial_x N = 1+y^2$

$\partial_y M \neq \partial_x N$. NOT exact! ← 3'

Multiplied by $\frac{1}{xy^3}$.

$$2x + \frac{1+y^2}{y^3}y' = 0$$

$M'(x,y) = 2x$. $N'(x,y) = \frac{1+y^2}{y^3}$ #

$\partial_y M' = 0$. $\partial_x N' = 0$

$\partial_y M' = \partial_x N'$ exact! ← 3'

Let $\begin{cases} \partial_x P(x,y) = M' = 2x. \\ \partial_y P = N' = \frac{1+y^2}{y^3} \end{cases}$ ← 4'

$\Rightarrow P(x,y) = x^2 - \frac{1}{2}\frac{1}{y^2} + \ln|y|$ ← 3'

Solution: $x^2 - \frac{1}{2y^2} + \ln|y| = C$ for arbitrary constant.

2'

5 #

5. Find the integrating factor and solve

$$1 + \left(\frac{x}{y} - \cos y\right) y' = 0.$$

Suppose multiplied by $\mu(x,y)$, the equation turns exact.

$$\mu(x,y) + \mu(x,y)\left(\frac{x}{y} - \cos y\right) y' = 0.$$

If it's exact then:

$$\partial_y \mu = \partial_y \left[\mu\left(\frac{x}{y} - \cos y\right)\right]$$
$$= \partial_y \mu \left(\frac{x}{y} - \cos y\right) + \mu x \left(\frac{1}{y}\right)$$

5'

One possible choice is let $\mu(x,y) = y$. 5'

then $\partial_y \mu = 1$, $\partial_x \mu = 0$.

Thus, $y + (x - y\cos y) y' = 0$.

Let $P(x,y)$ satisfies

8' $\begin{cases} \partial_x P = y \\ \partial_y P = x - y\cos y \end{cases} \Rightarrow \begin{cases} P = xy + C(y) \\ P = xy - y\sin y - \cos y + C(x) \end{cases}$

$\Rightarrow P = xy - y\sin y - \cos y$ 4'

So the solution is:

$xy - y\sin y - \cos y = C$. for arbitrary constant

3'

8 #